MATHEMATICS METHODS

MAWA Semester 2 (Unit 3&4) Examination 2018
Calculator-free

Marking Key

The release date for this exam and marking scheme is

- **the end of week 1 of term 4, Fri October 12th 2018**

Section One: Calculator-free (54 Marks)

Question 1 (a) (3 marks)

Solution	
$\int_1^4 \left(6x^2 + \frac{1}{2\sqrt{x}}\right) dx$ $= \left[2x^3 + \sqrt{x}\right]_1^4$ $= \left(2(64)+2\right) - \left(2+1\right) = 127$	
Mathematical behaviours	Marks
• integrates square root function correctly	1
• substitutes limits into correct anti-derivative	1
• evaluates result	1

Question 1 (b) (2 marks)

Solution	
$g'(x) \quad = e^{\frac{x+1}{2}}$ $g(x) \quad = 2e^{\frac{x+1}{2}} + c$ $(3, e^2) \Rightarrow e^2 = 2e^2 + c \Rightarrow c = -e^2$ $\therefore g(x) = 2e^{\frac{x+1}{2}} - e^2$	
Mathematical behaviours	Marks
• anti-differentiates correctly	1
• substitutes in $(3, e^2)$ to determine c	1

Question 1 (c) (2 marks)

Solution	
$\int_0^{\frac{\pi}{2}} \frac{d}{du} \sin u \; du = \left[\sin u\right]_o^{\frac{\pi}{2}} = 1$	
Mathematical behaviours	Marks
• applies the fundamental theorem	1
• evaluates result	1

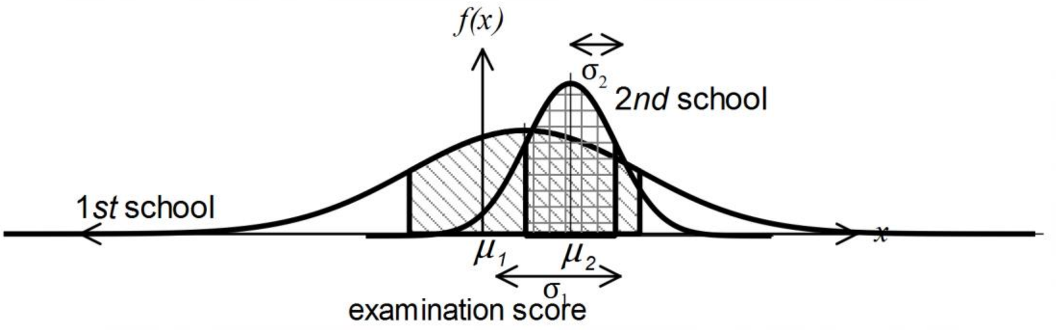
Question 2 (a) **(2 marks)**

Solution
$X \sim N(45, 9^2)$ $z_{63} = \frac{63-45}{9} = 2$ ie 63 represents 2 std deviations above the mean 2.5% of the population is above 63 $\therefore 0.025 \times 150 = 3.75$ ie approximately 4 students scored above Joanne.

Mathematical behaviours	Marks
• states that 63% represents 2 std deviations above the mean	1
• determines number of students above Joanne	1

Question 2 (b) **(2 marks)**

Solution



Mathematical behaviours	Marks
• diagram demonstrates that both distributions are normally distributed and $\mu_1 < \mu_2$	1
• diagram clearly depicts $\sigma_1 > \sigma_2$	1

Question 2 (c) **(3 marks)**

Solution
$\mu_X = 45, \quad \sigma_X = 9$ $\mu_Y = 55, \quad \sigma_Y = 6$ $Y = aX + b$ $a = \frac{6}{9} = \frac{2}{3}$ $55 = \frac{2}{3} \times 45 + b$ $25 = b$ $\therefore a = \frac{2}{3}, b = 25$

Mathematical behaviours	Marks
• uses standard deviations to determine a	1
• states equation needed to solve for b	1
• determines b value	1

Question 3 (a) **(2 marks)**

Solution

X	5	(-3)
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

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Mathematical behaviours	Marks
• correct entries for X values	1
• determines probabilities correctly	1

Question 3 (b) **(2 marks)**

Solution

$E(X) = 5 \times \frac{1}{4} + (-3) \times \frac{3}{4}$
$= \frac{5}{4} - \frac{9}{4}$
$= (-1)$

On average, Michael will lose $1 per toss

Mathematical behaviours	Marks
• determines expected gain correctly	1
• explains meaning of the negative value	1

Question 3 (c) **(2 marks)**

Solution

With a loss of $1 per toss, this is not a “fair” game.

A game is considered “fair” if Michael will, on the average, come out even.
That is, an expected gain of zero will define a “fair” game.

Mathematical behaviours	Marks
• states game is “not fair”	1
• valid explanation	1

Question 4 (a) **(3 marks)**

Solution
$16^x - 5\times 8^x = 0$ ie $2^{4x} = 5\times 2^{3x}$ ie $4x\log 2 = \log 5 + 3x\log 2$ · ie $x\log 2 = \log 5$ ie $x = \dfrac{\log 5}{\log 2}$

Mathematical behaviours	Marks
• rearranges equation and writes in exponential form	1
• applies log laws to each term of equation	1
• rearranges equation to arrive at result	1

Question 4 (b) **(3 marks)**

Solution
$5^{(2+\log_5 3)} + \log_{\frac{1}{5}} 125$ $= 5^2.5^{\log_5 3} + \log_{\frac{1}{5}}(\frac{1}{5})^{-3}$ $= 25\times 3 - 3$. $= 75 - 3$ $= 72$

Mathematical behaviours	Marks
• uses $a^m.a^n = a^{m+n}$ and $a^{\log_a b} = b$	1
• expresses $\log_{\frac{1}{5}} 125\ as\ \log_{\frac{1}{5}}(\frac{1}{5})^{-3}$, hence value of (-3)	1
• evaluates expression	1

Question 5 (a) **(3 marks)**

Solution	
$y = \ln\sqrt{3x - x^2}$ $= \frac{1}{2}\ln(3x - x^2)$ $\frac{dy}{dx} = \frac{1}{2}.\frac{1}{(3x-x^2)}.\frac{(3-2x)}{1} \quad = \quad \frac{(3-2x)}{2(3x-x^2)}$	
Mathematical behaviours	**Marks**
• expresses $y = \ln\sqrt{3x - x^2}$ as $y = \frac{1}{2}\ln(3x - x^2)$	1
• uses $\frac{d}{dx}\ln x = \frac{1}{x}$	1
• applies chain rule correctly and simplifies	1

Question 5 (b) **(3 marks)**

Solution	
$\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{1+\sin^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{2\sin x\cos x}{1+\sin^2 x}\,dx$ $= \ln\lvert 1 + \sin^2 x\rvert_0^{\pi/4}$ $= \ln\left\lvert 1 + \frac{1}{2}\right\rvert - \ln\lvert 1\rvert$ $= \ln\frac{3}{2}$ or $\ln 3$ - $\ln 2$	
Mathematical behaviours	**Marks**
• states anti-derivative of function with bounds	1
• substitutes in limits of integration correctly using $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	1
• evaluates result	1

Question 5 (c) **(4 marks)**

Solution	
Let $y = x\cos x$ $\frac{dy}{dx} = -x\sin x + \cos x$ Hence $\int \frac{dy}{dx}\,dx = \int(-x\sin x + \cos x)\,dx$ ie $x\cos x = -\int x\sin x\,dx + \int \cos x\,dx + c$ ie $x\cos x = -\int x\sin x\,dx + \sin x$ ie $\int x\sin x\,dx = \sin x - x\cos x + c$	
Mathematical behaviours	**Marks**
• states correct derivative	1
• integrates both sides	1
• applies Fundamental Theorem	1
• rearranges to arrive at correct result	1

Question 6 (a) **(2 marks)**

Solution	
$\hat{p}=1 \Rightarrow$ 5 heads in 5 tosses $\therefore$ probability = $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$	
Mathematical behaviours	**Marks**
• identifies that each toss must result in a head	1
• determines probability	1

Question 6(b) **(4 marks)**

Solution	
$\hat{p}$ is normally distributed with $\mu = 0.5$ and $\sigma=\sqrt{\frac{0.5\times 0.5}{100}} = .05$ $z_{0.55} = \frac{0.55-0.5}{0.05} = 1$ Hence, $P\left(\hat{p} > 0.55\right) = P(z>1) \approx 0.16$	
Mathematical behaviours	**Marks**
• identifies that $\hat{p}$ will be normally distributed	1
• determines mean and standard deviation for distribution of $\hat{p}$	1
• determines Z score associated with $\hat{p} = 0.55$	1
• determines probability	1

Question 6 (c) **(3 marks)**

Solution	
$P(\widehat{p_1} = \widehat{p_2}) = P(\widehat{p_1} = \widehat{p_2} = 0) + P\left(\widehat{p_1} = \widehat{p_2} = \frac{1}{3}\right) + P\left(\widehat{p_1} = \widehat{p_2} = \frac{2}{3}\right) + P(\widehat{p_1} = \widehat{p_2} = 1)$ (*) $= (P(\widehat{p_1} = 0)^2 + (P(\widehat{p_1} = 1/3)^2 + (P(\widehat{p_1} = 2/3)^2 + (P(\widehat{p_1} = 1)^2$ (**) $= \left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{20}{64} = \frac{5}{16}$	
Mathematical behaviours	**Marks**
• determines $\hat{p}$ values $0, \frac{1}{3}, \frac{2}{3}, 1$	1
• states calculation required to determine probability	1
• evaluates required sum	1

Question 7 (a) **(4 marks)**

Solution

$f'(x) = 3x^2e^{-x} - x^3e^{-x} = x^2(3-x)e^{-x}$

$f'(x) = 0 \Rightarrow x = 0$ or $x = 3$.

$f(0) = 0$, $f(3) = 3^3e^{-3}$, so f has stationary points at $(0,0)$ and at $(3, 3^3e^{-3})$

Since $f'(x) \geq 0$ if $x < 3$ and $f'(x) < 0$ if $x > 3$,
f has a point of inflection at $(0,0)$
and f has a local maximum at $(3, 3^3e^{-3})$

x	3^-	3	3^+
$f'(x)$	+ve	0	-ve

Mathematical behaviours	Marks
• differentiates correctly	1
• equates $f'(x) = 0$ and determines co-ordinates of stationary points	1
• justifies nature of first stationary point	1
• justifies nature of 2nd stationary point	1

Question 7 (b) **(2 marks)**

Solution

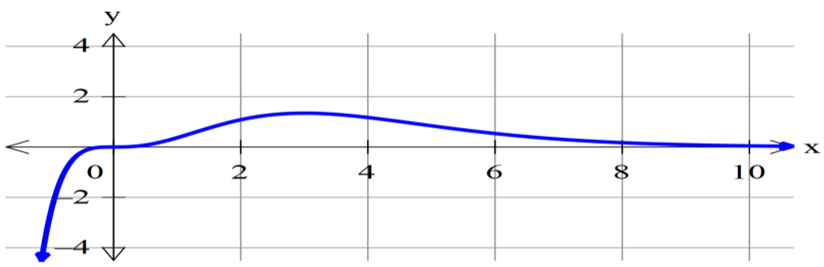
Yes.

Reason: $f(3) = 3^3e^{-3} = \left(\frac{3}{e}\right)^3 > 1$ since $0 < e < 3$

Mathematical behaviours	Marks
• gives correct answer	1
• gives a valid reason	1

Question 7 (c) **(3 marks)**

Solution



Mathematical behaviours	Marks
• shows inflection point at origin	1
• shows maximum at $x = 3$	1
• shows correct limits as $x \to \infty$ and $x \to -\infty$	1